so as to give a fine red-pink colour by transmitted light, even when so little chromium is present that the glassy bead is scarcely at all green. If too strongly heated the pink tint is lost. This compound is of interest in connexion with the colour of rubies and other minerals coloured red by chromic oxide. To others, like the emerald, it imparts a green colour, and on the whole it acts on light in such a variable manner according to the presence of other substances, that the spectra may be made use of as a means of identifying particular minerals, though they do not present anything like such striking anomalies as those met with in the compounds of zirconia with the oxides of uranium.

II. "On the Mathematical Theory of Stream-lines, especially those with four Foci and upwards." By WILLIAM JOHN MACQUORN RANKINE, C.E., LL.D., F.R.SS. Lond. and Edinb., &c. Received January 1, 1870.

(Abstract.)

A *Stream-line* is the line that is traced by a particle in a current of fluid. In a steady current each individual stream-line preserves its figure and position unchanged, and marks the track of a filament or continuous series of particles that follow each other. The motions in different parts of a steady current may be represented to the eye and to the mind by means of a group of stream-lines.

Stream-lines are important in connexion with naval architecture; for the curves which the particles of water describe relatively to a ship, in moving past her, are stream-lines; and if the figure of a ship is such that the particles of water glide smoothly over her skin, that figure is a *stream-line surface*, being a surface which contains an indefinite number of stream-lines.

The author in a previous paper proposed to call such stream-lines *Neoïds*; that is, ship-shape lines.

The author refers to previous investigations relating to stream-lines, and especially to those of Mr. Stokes, in the Cambridge Transactions for 1842 and 1850, on the "Motion of a Liquid past a Solid," and of Dr. Hoppe, on the "Stream-lines generated by a Sphere," in the Quarterly Journal of Mathematics for 1856, and to his own previous papers on "Plane Water-lines in Two Dimensions," in the Philosophical Transactions for 1864, and on "Stream-lines," in the Philosophical Magazine for that year. He states that all the neoïd or ship-shape stream-lines whose properties have hitherto been investigated in detail are either *unifocal* or *bifocal*; that is to say, they may be conceived to be generated by the combination of a uniform progressive motion, with another motion consisting in a divergence of the particles from a certain point or focus, followed by a convergence either towards the same point or towards a second point. Those which are

continuous closed curves when unifocal are circular, and when bifocal are blunt-ended ovals, in which the length may exceed the breadth in any given proportions. To obtain a unifocal or bifocal neoïd resembling a longitudinal line of a ship with sharp ends, it is necessary to take a part only of a stream-line, and then there is discontinuity of form and of motion at each of the two ends of that line.

The author states that the occasion of the investigation described in the present paper was the communication to him by Mr. William Froude of some results of experiments of his on the resistance of model boats, of lengths ranging from three to twelve feet. A summary of those results is printed at the end of a Report to the British Association on the "State of Existing Knowledge of the Qualities of Ships." In each case two models were compared together of equal displacement and equal length; the water-line of one was a wave-line with fine sharp ends, that of the other had blunt rounded ends, each joined to the midship body by a slightly hollow neck—a form suggested, Mr. Froude states, by the appearance of water-birds when swimming. At low velocities, the resistance of the sharp-ended boat was the smaller; at a certain velocity, bearing a definite relation to the length of the model, the resistances became equal, and at higher velocities the round-ended model had a rapidly increasing advantage over the sharp-ended model.

Hence it appeared to the author to be desirable to investigate the mathematical properties of stream-lines resembling the water-lines of Mr. Froude's bird-like models; and he has found that endless varieties of such forms, all closed curves free from discontinuity of form and of motion, may be obtained by using *four* foci instead of two. They may be called from this property *quadrifocal stream-lines*, or, from the idea that suggested such shapes to Mr. Froude, *cycnoïds*; that is, swan-like lines *.

Those lines are not to be confounded with the lines of a yacht having at a distance the appearance of a swan, which was designed and built some years ago by Mr. Peacock, for the figure of that vessel is simply oval.

The paper contains four chapters. The first three are mainly cinematical and geometrical, and relate to the forms of stream-line surfaces in two and in three dimensions, especially those with more than one pair of foci and surfaces of revolution, to the methods of constructing graphically and without calculation, by means of processes first applied to lines of magnetic force by Mr. Clerk Maxwell, the traces of such surfaces, which methods are exemplified by diagrams drawn to scale, and to the motions of the particles of liquid past those surfaces. The fourth chapter is dynamical: it treats of the momentum and of the energy of the disturbance in the liquid, caused by the progressive motion of a solid that is bounded by a ship-shape stream-line surface of any figure whatsoever; of the ratio borne by the total energy of the disturbance in the liquid to that of the disturbing body when that body displaces a mass of liquid equal to its own mass,

* Κυκνοειδής.

which ratio ranges in different cases from $\frac{1}{2}$ to 1 ; of the acceleration and
retardation of ships as affected by the disturbance in the water, and espe-
cially of the use of experiments on the retardation of ships in finding their
resistance ; and of the disturbances of pressure which accompany the
disturbances of motion in the liquid. Up to this point the dynamical
principles arrived at in the fourth chapter are certain and exact, like the
geometrical and cinematic principles in the three preceding chapters. The
results obtained in the remainder of the fourth chapter are in some respects
approximate and conjectural, and are to a great extent designed to suggest
plans for future experiments, and rules for their reduction. These results
relate to the disturbances of level which accompany the disturbances of
motion when the liquid has a free upper surface, to the waves which
originate in those disturbances of level, and the action of those waves in
dispersing energy and so causing resistance to the motion of the vessel,
to friction, or skin-resistance, and the "wake" or following current which
that kind of resistance causes the disturbing solid body to drag behind it,
and to the action of propelling instruments in overcoming different kinds
of resistance.

The resistance caused by viscosity is not treated of, because its laws have
been completely investigated by Mr. Stokes, and because for bodies of the
size of ships, and moving at their ordinary velocities, that kind of resistance
is inconsiderable compared with skin-resistance and wave-resistance. The
resistance caused by discontinuity of figure is stated to be analogous in its
effects to friction, but it is not investigated in detail, because ships ought
not to be built of discontinuous (commonly called "unfair") figures.

SUPPLEMENT.   Received January 8, 1870.

The author in the first place calls attention to the agreement between
the position of the points at which there is no disturbance of the pressure
on the surface of a sphere, as deduced from Dr. Hoppe's investigation,
published in 1856 (Quarterly Journal of Mathematics), and on the surface
of a short vertical cylinder with a flat bottom, as determined by the experi-
ments of the Rev. E. L. Berthon before 1850 (Proc. Roy. Soc. vol. v.
1850 ; also Transactions of the Society of Engineers, 6th December, 1869).
The theoretical value of the angular distance of those points from the
foremost pole of the sphere is $\sin^{-1}\frac{2}{3}=41° 49'$; the value deduced from
experiment is $41° 30'$.

The author then adds some remarks on a suggestion made by Mr.
William Froude, that the wave-resistance of a ship is diminished when two
series of waves originating at different points of her surface partially
neutralize each other by interference ; and states that, with regard to this
and many other questions of the resistance of vessels, a great advancement
of knowledge is to be expected from the publication in detail of the results
of experiments on which Mr. Froude has long been engaged.